

A Small Observation on Co-categories

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Abstract

Various concerns suggest looking for internal co-categories in categories with strong logical structure. It turns out that in any coherent category \mathcal{E} , all co-categories are co-equivalence relations.

Definition 1 Let \mathcal{E} be any category. An (internal) co-category \mathbf{Q} in \mathcal{E} is an internal category in \mathcal{E}^{op} , i.e. objects and morphisms in \mathcal{E}

$$Q^0 \begin{matrix} \xrightarrow{l} \\ \xleftarrow{r} \end{matrix} Q^1 \xrightarrow{q} Q^1 +_{Q^0} Q^1$$

such that the following diagrams commute:

$$\begin{array}{ccc} Q^0 & \xrightarrow{l} & Q^1 \xleftarrow{r} A^0 \\ \downarrow l & & \downarrow q \quad \downarrow r \\ Q^1 & \xrightarrow{\nu_1} & Q^1 +_{Q^0} Q^1 \xleftarrow{\nu_2} Q^1 \end{array} \quad \begin{array}{ccc} Q^1 & \xrightarrow{q} & Q^1 +_{Q^0} Q^1 \\ \downarrow q & & \downarrow [q, \nu_3] \\ Q^1 +_{Q^0} Q^1 & \xrightarrow{[\nu_1, q]} & Q^1 +_{Q^0} Q^1 +_{Q^0} Q^1 \end{array}$$

$$\begin{array}{ccc} Q^0 & \xrightarrow{l} & Q^1 \xleftarrow{r} Q^0 \\ & \searrow l & \downarrow i \quad \swarrow r \\ & & Q^0 \end{array} \quad \begin{array}{ccc} & Q^1 & \\ 1 \swarrow & \downarrow q & \searrow 1 \\ Q^1 & \xleftarrow{[li, 1]} & Q^1 +_{Q^0} Q^1 \xrightarrow{[1, ri]} Q^1 \end{array}$$

Definition 2 A co-category \mathbf{Q} is a co-preorder if the maps l, r are jointly epimorphic.

A co-category \mathbf{Q} is a co-groupoid if there is a map $s : Q^1 \rightarrow Q^1$ satisfying the duals of the usual identities for the inverse map of a groupoid.

A co-groupoid \mathbf{Q} is a co-equivalence relation if it is a co-preorder.

Remark 1 In a co-preorder, the co-composition q is uniquely determined by the maps l, r, i ; likewise, in a co-groupoid, the co-inverse map s is determined by the rest of the structure.

Together with the obvious maps, these give categories and full inclusions

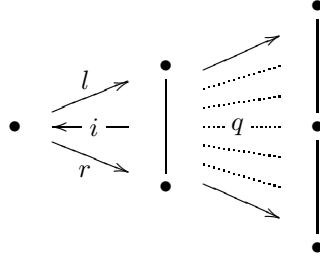
$$\mathbf{CoEqRel}(\mathcal{E}) \hookrightarrow \mathbf{CoPreOrd}(\mathcal{E}) \hookrightarrow \mathbf{CoCat}(\mathcal{E}).$$

Example If \mathcal{E} has all (or enough) pushouts and $m : S \rightarrowtail A$ is any monomorphism, then the *co-kernel pair* of m is a co-equivalence relation

$$A \begin{array}{c} \xrightarrow{\nu_1} \\ \xRightarrow{[1_A, 1_A]} \\ \xleftarrow{\nu_2} \end{array} A +_S A \xrightarrow{[\nu_1, \nu_3]} A +_S A +_S A.$$

This gives the object part of a functor $\mathbf{Mono}(\mathcal{E}) \rightarrow \mathbf{CoEqRel}(\mathcal{E})$, which (almost by definition) is one half of an equivalence whenever \mathcal{E} is co-exact. (Here $\mathbf{Mono}(\mathcal{E})$ denotes the full subcategory of $\mathcal{E}^{\rightarrow}$ on monomorphisms.)

Example A paradigmatic example is the interval \mathbf{I} in \mathbf{Top} , where I^0 is a singleton, I^1 is the unit interval, l and r are the endpoints, $I^1 +_{I^0} I^1$ is two copies of the interval joined end to end, and q is the obvious “stretching” map. Unfortunately, this is also of course not an actual co-category — the axioms hold only up to homotopy. However, it provides a very useful mental picture for the arguments below; and if we delete the interior of the interval, we obtain a genuine co-category. See also the examples below for more versions of the interval.



Definition 3 A coherent category is a category with all finite limits, and images and unions that are stable under pullback.

[4, A1.3–4] gives various basic results on coherent categories, which we will use here without comment.

Definition 4 Coherent logic is the fragment of first-order logic built up from atomic formulæ using finite con-/dis-junction and existential quantification.

Coherent logic is discussed in [4, D1.1–2]; the essential point is that coherent logic can be interpreted soundly in coherent categories, and so may be used as an internal language for working in them.

Proposition 1 In a coherent category \mathcal{E} , every co-category \mathbf{Q} is a co-equivalence relation.

Proof First, we show that any co-category \mathbf{Q} is a co-preorder.

Arguing in the internal logic: given x in Q^1 , consider $q(x)$, in $Q^1 +_{Q^0} Q^1$. Either there is some y in Q^1 with $q(x) = \nu_1(y)$, or else some y with $q(x) = \nu_2(y)$. In the first case, we then have $x = [li, 1]q(x) = li(y)$; in the second, $x = ri(y)$. Thus any x in Q^1 is in the image of either l or r , i.e. l and r are jointly covering, hence epi. (Indeed, in the first case $x = li(y) = l(il)i(y) = li(li(y)) = li(x)$, and in the second, $x = ri(x)$.)

Restating this diagrammatically: $Q^1 +_{Q^0} Q^1$ is the union of the subobjects $\nu_j : Q^1 \rightarrow Q^1 +_{Q^0} Q^1$, so Q^1 is the union of the subobjects $m_j = q^*(\nu_j)$:

$$\begin{array}{ccc} P_j & \xrightarrow{q_j} & Q^1 \\ \downarrow m_j & & \downarrow \nu_j \\ Q^1 & \xrightarrow{q} & Q^1 +_{Q^0} Q^1 \end{array}$$

In particular, the pair m_1, m_2 are jointly covering. But by the co-unit identities, $liq_1 = [li, 1]\nu_1q_1 = [li, 1]qm_1 = m_1$, and similarly $riq_2 = m_2$. Thus liq_1, riq_2 are jointly covering, and hence so are l, r .

Now, we check that any co-preorder is a co-equivalence relation. (We give only the diagrammatic version. Exercise: restate this in the internal logic!) We want to define $s : Q^1 \rightarrow Q^1$ with $sl = r$, $sr = l$. Since l, r are monos with union Q^1 , the pullback square

$$\begin{array}{ccc} \bullet & \xrightarrow{\pi_2} & Q^0 \\ \pi_1 \downarrow & & \downarrow r \\ Q^0 & \xrightarrow{l} & Q^1 \end{array}$$

is also a pushout, so to construct s as above, it is enough to show that $r\pi_1 = l\pi_2$. But $\pi_1 = il\pi_1 = ir\pi_2 = \pi_2$, so $r\pi_1 = r\pi_2 = l\pi_1 = l\pi_2$, and we are done. \square

Corollary 2 *If \mathcal{E} is a coherent category with co-kernel pairs of monos, then $\mathbf{CoCat}(\mathcal{E}) \simeq \mathbf{Mono}(\mathcal{E})$. (In particular, this holds if \mathcal{E} is a pretopos [4, A1.4.8].)*

Proof A coherent category is certainly co-effective, so if it has co-kernel pairs, it is co-exact. \square

Corollary 3 *For any topos \mathcal{E} , $\mathbf{CoCat}(\mathcal{E}) \simeq (\mathcal{E}/\Omega)_{colax}$.* \square

(A *colax* map $(X, \varphi) \rightarrow (Y, \psi)$ is a map $f : X \rightarrow Y$ such that $\varphi \leq_\Omega \psi f$.)

In particular, inspecting this equivalence, we see that in this case there is a universal internal co-category in \mathcal{E} , from which every co-category in \mathcal{E} may be obtained uniquely by pullback: it is the co-kernel pair of $\top : 1 \rightarrow \Omega$.

Example The condition that unions are preserved by pullback is crucial: \mathbf{AbGp} , for instance, is regular, and has unions, but there is a non-co-preorder co-category corresponding to the interval pictured above, given by the objects

$$Q^0 = \langle v_0 \rangle \quad Q^1 = \langle v_0, e_1, v_1 \rangle \quad Q^1 +_{Q^0} Q^1 = \langle v_0, e_1, v_1, e_2, v_2 \rangle$$

(with the natural maps making this a pushout), and maps given by the matrices

$$l = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad r = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad i = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \quad q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This example may be given more structure; it is, for instance, the total space of an natural co-category in $\mathbf{Ch}(\mathbf{AbGp})$. Since all the underlying groups are free and of finite rank, dualising by transposing matrices also gives corresponding categories in \mathbf{AbGp} and $\mathbf{Ch}(\mathbf{AbGp})$.

However, any category in a Mal'cev category is a groupoid (this has been observed by various authors, e.g. in [3]), so any co-category in a co-Mal'cev category (e.g. in an Abelian category, or a topos [2]) is a co-groupoid.

Example An example of a non-co-groupoid co-category is the interval \mathbf{I} in \mathbf{Cat} , with $I^0 = (\cdot)$, $I^1 = (\cdot \rightarrow \cdot)$; seen as a co-simplicial object, this is just the usual inclusion functor $\Delta \hookrightarrow \mathbf{PreOrd} \hookrightarrow \mathbf{Cat}$.

Indeed, the functor $\mathbf{Cat} \rightarrow \mathbf{SSet} \rightarrow \mathbf{SAbGp} \rightarrow \mathbf{Ch}(\mathbf{AbGp}) \rightarrow \mathbf{Ch}(\mathbf{AbGp})$ “take nerve; take free abelian groups; normalise to a complex; quotient out by subcomplex generated in degrees ≥ 2 ” sends \mathbf{I} to the co-category in $\mathbf{Ch}(\mathbf{AbGp})$ of the previous example.

Co-categories arise as candidate “interval objects” when using 2-categories to model intensional type theory [1]. There, one seeks them in categories with some sort of “weakened” logical structure; the present result confirms the suspicion that examples in classical “strict” logical categories are necessarily fairly trivial.

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References

- [1] Steve Awodey and Michael A. Warren, *Homotopy-theoretic models of identity types*, Math. Proc. Cambridge Philos. Soc. 146 (2009), no.1, pp.45–55, [arXiv:0709.0248](#) [math.LO]
- [2] Dominique Bourn, *Mal'cev Categories and Fibration of Pointed Objects*, Applied Categorical Structures, Vol.4 (1996), pp.307–327
- [3] A. Carboni, G.M. Kelly, M.C. Pedicchio *Some remarks on Maltsev and Goursat categories*, Applied Categorical Structures, Vol.1 (1993), pp.385–421
- [4] Peter Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, Oxford University Press (2002)